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Holographic Duals of Flavored $\mathcal{N} = 1$ Super Yang-Mills: Beyond the Probe Approximation

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Abstract

We construct backreacted D3/D7 supergravity backgrounds which are dual to four-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Yang-Mills at large N_c with flavor quarks in the fundamental representation of $\mathrm{SU}(N_c)$. We take into account the backreaction of D7-branes on either $\mathrm{AdS}_5 \times S^5$ or $\mathrm{AdS}_5 \times T^{1,1}$, or more generically on backgrounds where the space transverse to the D3-branes is Kähler. The construction of the backreacted geometry splits into two stages. First we determine the modification of the six-dimensional space transverse to the D3 due to the D7, and then we compute the warp factor due to the D3.

The $\mathcal{N} = 2$ background corresponds to placing a single stack of N_f D7-branes in $\mathrm{AdS}_5 \times S^5$. Here the Kähler potential is known exactly, while the warp factor is obtained in certain limits as a perturbative expansion. By placing another D7$'$ probe in the backreacted D3/D7 background, we derive the effect of the D7-branes on the spectrum of the scalar fluctuations to first order in N_f. The two systems with $\mathcal{N} = 1$ supersymmetry that we discuss are D3/D7/D7$'$ and D3/D7 on the conifold. In both cases, the Kähler potential is obtained perturbatively in the number of D7-branes. We provide all the ingredients necessary for the computation of each term in the expansion, and in each case give the first few terms explicitly. Finally, we comment on some aspects of the dual gauge theories.

1 Introduction

In the framework of the gauge/gravity correspondence [1] a complete holographic description of QCD is still beyond our current reach. Nevertheless, substantial progress has been achieved in the holographic understanding of pure $\mathcal{N} = 1$ super Yang-Mills (SYM) at large N_c [2, 3]. The holographic understanding of flavor dynamics at the quantitative level has developed slower, in great measure due to the lack of a fully backreacted supergravity background describing gauge theories with fundamental flavor. In this paper we address the construction of fully backreacted supergravity backgrounds dual to $\mathcal{N} = 1$ SYM with fundamental flavor.

Recently, there has been much attention devoted to flavor dynamics in the probe approximation limit [4–16]. However, a significant limitation of the probe approximation is that in that regime the number of flavors is necessarily much smaller than the number of colors: $N_f/N_c \ll 1$. Still, even in the probe approximation, there have been some quite interesting results [17, 18].

Why go beyond the probe approximation? One of the major stepping stones in a holographic understanding of QCD is a control of flavor dynamics. For example, dynamical mesons and baryons play a central rôle in obtaining a complete understanding of the QCD phase diagram. In particular, in the large N_c limit the critical temperature should behave very differently in the presence of baryons. On the other hand, it is known that baryons play a subleading rôle at large N_c, which leads to the question of how precisely they change the critical temperature. On a more formal level, the quantum moduli space of SU(N_c) supersymmetric gauge theories with N_f flavors crucially depends on the relative value of N_c and N_f. There is a host of phenomena in flavor dynamics of supersymmetric theories, such as the conformal window, that can be explored only in the limit of comparable N_c and N_f. Having $N_f \ll N_c$ excludes, in principle, the possibility of a holographic description of such phenomena. Clearly, unless a fully backreacted solution becomes available, a quantitative understanding of several physical questions of gauge theories will remain elusive in the holographic approach.

In this paper we address the question of backreaction in the context of supergravity backgrounds dual to $\mathcal{N} = 1$ supersymmetric gauge theories. While the general class of supergravity backgrounds dual to $\mathcal{N} = 1$ gauge theories is wide, we concentrate only on backgrounds containing D3 and D7-branes. In this case the backreacted metric of

the space transverse to the stack of D3-branes remains Kähler. For Kähler metrics a remarkable simplification of the Einstein equations takes place. In fact, solving the Einstein equations is equivalent to solving an equation that in many cases becomes a standard Monge-Ampère equation. Therefore, the problem of finding backreacted solutions splits into two main steps. In the first one, after finding a suitable Ansatz for the dilaton, we determine the Kähler potential of the resulting six-dimensional background transverse to the D3 worldvolume. In the second step we find the warp factor due to the presence of the D3-branes.

The paper is organized as follows. In section 2 we review the construction of the D3/D7 system to introduce notation and comment on some of the choices that we make further in the paper. Section 3 contains a description of the D3/D7/D7′ system, that is, two perpendicular stacks of the D7-branes that are parallel to the D3. This system has $\mathcal{N}=1$ supersymmetry, and we discuss it in some detail. Section 4 presents a probe calculation of the scalar (meson) spectrum of the $\mathcal{N}=1$ gauge theory. We provide the spectrum as an expansion in the number of D7-branes of the D3/D7 geometry. We then turn to the conifold picture by first reviewing the structure of the conifold as a Kähler manifold in section 5, and then describing a solution with D7-branes holomorphically embedded in the conifold in section 6. Section 7 contains some comments on the gauge theory side. We conclude in section 8 by enumerating a number of open problems. Appendix A contains some useful results pertaining to properties of the Monge-Ampère equation.

2 D3/D7 Systems

In this section we review the basic setup for examining D3/D7 systems. The D3-brane geometry by itself breaks the 10-dimensional Poincaré invariance into manifest $\mathrm{SO}(3,1)\times\mathrm{SO}(6)$. This may be represented by a metric of the form

$$ds_{10}^2 = h^{-1/2}(y_m)dx_\mu^2 + h^{1/2}(y_m)g_{mn}dy^m dy^n, \tag{2.1}$$

where x^μ ($\mu=0,\ldots,3$) are flat coordinates on the longitudinal spacetime, and y^m ($m=4,\ldots,9$) are coordinates transverse to the D3-brane. The D7-branes are embedded in this 10 dimensional space such that they share the four longitudinal directions with the D3-branes while wrapping or spanning four out of the six transverse directions. This embedding may be represented pictorially as

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	·	·	·	·	·	·
D7	–	–	–	–	–	–	–	–	·	·

for the case of a single stack of D7-branes.

D7-brane configurations have been effectively studied in [19] in the context of cosmic strings, and D3/D7 systems have been examined in [20, 21]. The starting point for exploration of D3/D7 geometries preserving some fraction of supersymmetries is the set of Killing spinor equations obtained from the IIB supersymmetry transformations [22]

$$\begin{aligned}\delta\lambda &= i\gamma^M P_M \epsilon^* - \frac{i}{24}\gamma^{MNK} G_{MNK}\epsilon, \\ \delta\psi_M &= \left(D_M + \frac{i}{480}\gamma^{M_1\ldots M_5}\gamma_M F_{M_1\ldots M_5}\right)\epsilon - \frac{1}{96}\left(\gamma_M{}^{NPQ} - 9\delta_M^N\gamma^{PQ}\right) G_{NPQ}\epsilon^*, \end{aligned} \tag{2.2}$$

where $D_M = \nabla_M - \frac{i}{2}Q_M$ and

$$P_M = \frac{i}{2}\frac{\partial_M \tau}{\tau_2}, \qquad Q_M = -\frac{\partial_M \tau_1}{2\tau_2}, \tag{2.3}$$

are the $\mathrm{SL}(2,\mathbb{R})$ scalar kinetic and composite connection terms for the dilaton/axion pair $\tau = \tau_1 + i\tau_2$.

The Killing spinor equations simplify in the absence of G_3 flux to yield

$$\begin{aligned}\gamma^M P_M \epsilon^* &= 0, \\ \left(D_M + \frac{i}{480}\gamma^{M_1M_2M_3M_4M_5} F_{M_1M_2M_3M_4M_5}\right)\epsilon &= 0. \end{aligned} \tag{2.4}$$

Corresponding to the $4+6$ coordinate split of (2.1), we decompose the Dirac matrices according to $\gamma^\mu = (\Gamma^\mu \otimes 1)$, $\gamma^m = (\Gamma^5 \otimes \Gamma^m)$, where Γ represents either the $\mathrm{SO}(1,3)$ or $\mathrm{SO}(6)$ Dirac matrices as appropriate. This allows us to define the chirality matrices

$$\begin{aligned}\gamma^5_{(4)} &= i\gamma^0\cdots\gamma^3 = (\Gamma^5 \otimes 1), \\ \gamma^7_{(6)} &= -i\gamma^4\cdots\gamma^9 = (1 \otimes \Gamma^7). \end{aligned} \tag{2.5}$$

The complex IIB spinor ϵ has definite chirality, $\gamma^{11}\epsilon = \epsilon$. Hence the four and six-dimensional chiralities are correlated, so that $\gamma^5_{(4)}\epsilon = \gamma^7_{(6)}\epsilon = s\epsilon$ where $s = \pm 1$. In addition, the coordinate split also allows us to write the most general five form consistent with four-dimensional Poincaré symmetry and self-duality:

$$F_{M_1M_2M_3M_4M_5} = -\epsilon^{(6)}_{M_1M_2M_3M_4M_5M_6} F^{M_6} + 5\epsilon^{(4)}_{[M_1M_2M_3M_4} F_{M_5]}, \tag{2.6}$$

where F^M is a function only of the y^m coordinates, and is zero for $M = 0 \dots 3$. To insure Poincaré symmetry we must also take τ to be a function only of the y^m coordinates.

Using the above, the Killing equations, (2.4), can now be written as

$$
\begin{aligned}
P_m (1 \otimes \Gamma^m) \epsilon^* &= 0, \\
\partial_\mu \epsilon - \left(\frac{s}{8} \partial_n \log(h) + \frac{1}{2} F_n \right) (\Gamma_\mu \otimes \Gamma^n) \epsilon &= 0, \\
\nabla_m^{(6)} \epsilon - \frac{s}{2} F_m \epsilon + \left(\frac{1}{8} \partial_n \log(h) + \frac{s}{2} F_n \right) (1 \otimes \Gamma_m{}^n) \epsilon - \frac{i}{2} Q_m \epsilon &= 0,
\end{aligned}
\tag{2.7}
$$

where $\nabla^{(6)}$ is covariant with respect to the metric on the six dimensional space transverse to the D3 branes. Taking ϵ to be only a function of y^m, the second equation can immediately be solved by taking $\gamma^5_{(4)} \epsilon = \epsilon$ (so that $s = 1$) and at the same time setting

$$
F_m = -\frac{1}{4} \partial_m \log(h) = -\frac{\partial_m h}{4h}. \tag{2.8}
$$

This is a remarkable result, as it demonstrates that the usual relation between the D3-brane warp factor $h(y^m)$ and the five-form remains unchanged, even in the presence of D7-branes. Note that the $s = 1$ condition on Killing spinors yields the familiar result that the D3-branes by themselves are half-BPS objects.

Taking the warp factor as above, the Killing spinor equations (2.7) reduce to

$$
\gamma^m P_m \eta^* = 0, \qquad (\nabla_m - \frac{i}{2} Q_m) \eta = 0, \tag{2.9}
$$

where $\eta = h^{1/8} \epsilon$. These equations may be used to determine the structure of the D7-branes. Since the second equation above implies that η is covariantly constant, the connection ∇_m is in U(3), and the transverse space to the D3-branes is complex Kähler. The first of the above equations is a familiar one for D7-branes, namely $(\gamma^m \partial_m \tau) \eta^* = 0$, and can be satisfied by taking τ to be holomorphic on the complex 3-fold. This yields the further conditions $\gamma^m \eta^* = 0$ on the Killing spinors whenever $\partial_m \tau \neq 0$ (for the generic case), where z_m ($m = 1, 2, 3$) is now a complex coordinate. As a result, the D3/D7 system preserves 1/4, 1/8 and 1/8 of the IIB supersymmetries for τ depending holomorphically on one, two and three of the complex z_m coordinates, respectively. Note that there is no further reduction of supersymmetry for the last case, as the product of the three independent D7 projections gives precisely the D3 projection.

The second equation in (2.9) has a useful integrability condition, $R^{(6)}_{mn} = P_m P^*_n + P^*_m P_n$. This has a particularly nice form for Kähler geometry, namely $R^{(6)}_{m\bar{n}} = P_m P^*_{\bar{n}}$,

where m and $\bar{n}$ denote complex indices. Taking $\tau(z_m)$ to be holomorphic and using P_m from (2.3) results in

$$-\partial_m \partial_{\bar{n}} \log(\det(g^{(6)})) = -\partial_m \partial_{\bar{n}} \log(\tau - \tau^*), \tag{2.10}$$

which may be integrated to give

$$\det(\partial_m \partial_{\bar{n}} \mathcal{F}) = \Omega \bar{\Omega} \, \mathrm{Im}\,(\tau). \tag{2.11}$$

Here $\mathcal{F}$ is the Kähler potential and Ω is an arbitrary holomorphic function accounting for the integration factors. The same Ω is used for the holomorphic and antiholomorphic integration functions to insure that the Kähler potential is real. In cases with 'radial' symmetry, Eq. (2.11) can be reduced to a real Monge Ampère equation (see Appendix A for further comments on this equation).

Finally, returning to the D3-brane warp factor, the integrability condition combined with the Einstein equations gives the result that the function h is harmonic

$$\Box_{(6)} h = -(2\pi)^4 N_c \det(g^{(6)})^{-1/2} \delta^6 \left(y^m - y^m_{\mathcal{D}3} \right). \tag{2.12}$$

As a result, solving for the D3/D7 geometry can be reduced to several steps. First, we take a (noncompact) Calabi-Yau background geometry, and then turn on the D7-branes by choosing a holomorphic τ with appropriate monodromies. Then we must solve the complex Monge Ampère equation (2.11) for the Kähler potential, yielding the geometry of the six-dimensional transverse space. Finally, to include the D3-branes, we must solve the harmonic function equation (2.12). This gives the warp factor and corresponding five-form flux through (2.8). Unlike for most intersecting brane configurations, at least in principle this procedure yields fully localized D3/D7 solutions, although the resulting equations are often quite intractable in practice. Note that the possibility of constructing localized intersecting solutions for the D3/D7 system is a feature shared with the D2/D6 system [23].

2.1 D3 and a single stack of D7 in flat space

We now consider the simplest case where the six-dimensional transverse space to the D3-branes is flat before turning on any D7-branes. In particular

$$ds_{6,\text{ background}} = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3, \qquad \mathcal{F}_{\text{flat, background}} = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3. \tag{2.13}$$

In the presence of D7-branes, the Kähler function may be obtained by solving (2.11). However, before doing so, we must understand the form of the dilaton/axion field, $\tau(z_1, z_2, z_3)$.

For a single stack of D7-branes transverse to z_3, we may follow the procedure given in [19], which we outline here. If one requires modular invariance for τ, holomorphicity and the proper monodromy around the D7 branes, the basic solution can be written in terms of the modular invariant j-function as [19, 21]

$$j(\tau) = f(z_3), \tag{2.14}$$

where for now f is an arbitrary holomorphic function. In [19] it was observed that poles in f correspond to D7-branes, in the sense of decompactification at the cores of the branes (as will be indicated below). Solving (2.11) with this form of the dilaton/axion is quite involved; hence we only consider the neighborhood close to one of the poles

$$j(\tau) = \frac{b}{(z_3 - z_0)^{N_f}}. \tag{2.15}$$

The rôle of the j-function in the above expression is to provide a map from the fundamental domain into the whole complex plane which is then related to N_f coverings of the z_3-plane.

We have limited ourselves to z_3 close to z_0 which requires that τ_2 is large. In this case the j function can be approximated as $\exp(-2\pi i\tau)$, so that the above equation becomes

$$\tau(z_3) = i\left(\frac{\log(b)}{2\pi} - \frac{N_f}{2\pi}\log(z_3)\right), \qquad (z_3 \ll 1), \tag{2.16}$$

where we have shifted z_3 to remove z_0. This is known as the "decompactification limit" from F-theory because it corresponds to the modulus of the compactified torus going to infinity. We will work in this limit for the remainder of the paper. (Note that Eq. (2.16) is no longer appropriate for $z_3 \gtrsim 1$, where instead τ approaches a constant value.) It is also natural to consider scalings of z_3 so that b may be removed as well. Two scalings that are natural are $b = 1$ and $b = e^{2\pi}$, making the constant $\log(b)/2\pi$ either 0 or 1, respectively.

In [19], Ω is fixed by considering modular invariance of the metric, imposing regularity of the metric, and noting that $g_{3\bar{3}}$ is proportional to τ_2. This gives

$$g_{3\bar{3}} \sim \tau_2|\eta(\tau)|^4|z_3|^{-N_f/6}. \tag{2.17}$$

Near the core of the D7-brane, where τ_2 is large, this has the simple form

$$g_{3\bar{3}} \sim \tau_2. \tag{2.18}$$

In particular, the additional factor $|\eta(\tau)|^4 |z_3|^{-N_f/6}$ in the metric simply approaches a constant near the cores of the D7-branes.

We now turn to the question of the Kähler potential. We have chosen to stack the D7-branes together so that they are extended along z_1 and z_2, and are transverse in z_3. Making use of the background symmetry, and noting that τ_2 is a function of the real combination $z_3\bar{z}_3$ suggests that we take

$$\mathcal{F}_{D7} = z_1\bar{z}_1 + z_2\bar{z}_2 + f(z_3\bar{z}_3). \tag{2.19}$$

The background Kähler potential is obtained from (2.11) by taking $\operatorname{Im}\tau = 1$ and $\Omega = 1$, so that when $f = z_3\bar{z}_3$ we simply get flat space. We now take $b = e^{2\pi}$ and turn on non-zero N_f, so that (2.11) becomes

$$\partial_3\partial_{\bar{3}} f(z_3\bar{z}_3) \equiv (z_3\bar{z}_3)f'' + f' = \left(1 - \frac{N_f}{4\pi}\log(z_3\bar{z}_3)\right). \tag{2.20}$$

By making a change of variables, $y_3 = \log(z_3\bar{z}_3)$, we arrive at

$$f''(y_3) = \left(1 - \frac{N_f}{4\pi} y_3\right) e^{y_3}. \tag{2.21}$$

This may be straightforwardly integrated to obtain the Kähler function for a single stack of D7-branes located at $z_3 = 0$

$$\mathcal{F}_{\mathrm{D7}} = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - \frac{N_f}{4\pi}(z_3\bar{z}_3)(\log(z_3\bar{z}_3) - 2) + C_1\log(z_3\bar{z}_3) + C_2. \tag{2.22}$$

The constants C_1 and C_2 do not affect the metric; we will therefore set them to zero (we may always add a holomorphic plus antiholomorphic function to a Kähler potential). Of course, the transverse space metric is given by

$$ds_6^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + \left(1 - \frac{N_f}{4\pi}\log z_3\bar{z}_3\right) dz_3 d\bar{z}_3, \tag{2.23}$$

and this could have been obtained directly from (2.18) without even having to solve for the Kähler function. Nevertheless, this exercise in obtaining $\mathcal{F}$ by solving (2.11) will prove essential when considering multiple stacks of D7-branes as well as D7-branes in other backgrounds.

2.2 The warp factor

Now that we have found the D7-brane contribution to the transverse metric, (2.23), we may turn to the D3-brane warp factor. To determine the warp factor, we must solve the transverse Laplacian (2.12). For a single stack of D7-branes, we first rewrite the metric (2.23) in the generic form

$$ds_6^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + e^{\Psi(z_3,\bar{z}_3)} dz_3 d\bar{z}_3, \tag{2.24}$$

in which case the transverse Laplacian is more explicitly

$$\Box_{(6)} h = (\partial_1 \partial_{\bar{1}} + \partial_2 \partial_{\bar{2}} + e^{-\Psi} \partial_3 \partial_{\bar{3}}) h. \tag{2.25}$$

This expression for the warp factor was examined in [24], where it was solved to a first order approximation away from the D7-branes, where the metric is approximately flat, with a deficit angle.

In the absence of D7-branes, it is natural to place the stack of D3-branes at the origin, $z_m = 0$, in which case the SO(6) R-symmetry is preserved, so that $h = L^4/R^4$ where R is the radial coordinate, $R^2 = |z_1|^2 + |z_2|^2 + |z_3|^2$. In orienting the D7-branes to lie fully in the z_1 and z_2-planes, we preserve a natural $\mathrm{SO}(4) \times \mathrm{SO}(2) \subset \mathrm{SO}(6)$ subgroup. Note, however, that separating the D3 and D7-branes in the z_3-plane will break the SO(2). With this symmetry in mind, we introduce real coordinates

$$r^2 = |z_1|^2 + |z_2|^2, \qquad z_3 = \rho_3 e^{i\phi_3}, \tag{2.26}$$

so that

$$\Box_{(6)} h = \left(\frac{1}{r^3} \partial_r r^3 \partial_r + e^{-\Psi(\rho_3,\phi_3)} \left(\frac{1}{\rho_3} \partial_{\rho_3} \rho_3 \partial_{\rho_3} + \frac{1}{\rho_3^2} \partial_{\phi_3}^2 \right) \right) h(r, \rho_3, \phi_3). \tag{2.27}$$

If the D7-branes are separated by a distance d along the real z_3-axis, the metric function e^Ψ has the form

$$e^\Psi = \left(1 - \frac{N_f}{4\pi} \log(\rho_3^2 - 2\rho_3 d \cos\phi_3 + d^2) \right). \tag{2.28}$$

Although no straightforward solutions to (2.27) appear to exist in the general case, we may find approximate solutions in appropriate limits. For example, when $d = 0$ the D3-branes overlap the D7-branes, and the remaining SO(2) symmetry is restored. In this case, the Laplacian reduces to one with only two radial variables, ρ_3 and r. Furthermore, we may consider solving for the warp factor near the core of the D7-branes, where $e^\Psi \gg 1$. This is the decompactification region of the complex z_3-plane, and corresponds to the region where the backreaction is strong.

It can be checked that the equation[1] 2.27 allows separation of variables in the form:

$$h = f(r)g(\rho,\phi)\,Y^l(S^3), \tag{2.29}$$

where $Y^l(S^3)$ are spherical harmonics on S^3

$$\nabla^i\nabla_i\,Y^l = -l(l+2)\,Y^l. \tag{2.30}$$

The equation for the function $f(r)$ can be solved in terms of Bessel functions. The equation for g can be further separated in variables, after Fourier expanding in an appropriate angular variable, resulting in an equation for the radial variable along. The latter is a more involved second order differential equation which we will not discuss here in detail.

3 D3 and two stacks of D7-branes

The D3 geometry in the presence of a single stack of D7-branes is perhaps the most straightforward to investigate, as there is little ambiguity in how the D7-branes are turned on. However, we are generally more interested in configurations admitting $\mathcal{N}=1$ duals with flavor. This may be achieved by either starting with a transverse space to the D3-branes with reduced supersymmetry or by turning on multiple stacks of D7-branes. In this section we focus on the latter possibility by turning on two stacks of mutually perpendicular D7-branes.

In particular, we can construct a solution representing two stacks of D7-branes by placing, as before, N_3 D7-branes perpendicular to the z_3-plane and another set of N_2 D7-branes transverse to the z_2-plane. This configuration may be represented as follows

	0	1	2	3	4	5	6	7	8	9
D3	–	–	–	–	·	·	·	·	·	·
D7	–	–	–	–	–	–	–	–	·	·
D7′	–	–	–	–	–	–	·	·	–	–

Note that turning on a third stack of D7-branes transverse to z_1 would not further reduce any supersymmetry. However for simplicity we avoid this case as well as other possible non-perpendicular configurations.

[1] We are thankful to A. Hashimoto and M. Mahato for discussions relevant to this topic.

In order to have two stacks of D7-branes, located at $z_2 \to 0$ and $z_3 \to 0$, we take $\tau(z_1, z_2, z_3)$ to have the appropriate monodromies around the respective cycles. More precisely, near the D7-branes we have

$$\tau = i\left(1 - \frac{N_2}{2\pi}\log z_2 - \frac{N_3}{2\pi}\log z_3\right). \tag{3.1}$$

As a result, it is natural to take an Ansatz for the Kähler function in the form

$$\mathcal{F}_{\mathrm{D7/D7'}} = z_1\bar{z}_1 + f(z_2\bar{z}_2, z_3\bar{z}_3). \tag{3.2}$$

Since both $\mathcal{F}_{\mathrm{D7/D7'}}$ and $\mathrm{Im}\,\tau$ depend only on the magnitudes of the complex coordinates, we make a similar change of variables as in the single stack case

$$y_2 = \log(z_2\bar{z}_2), \qquad y_3 = \log(z_3\bar{z}_3), \tag{3.3}$$

so that the equation determining the Kähler potential, (2.11), becomes

$$f_{22}f_{33} - {f_{23}}^2 = \left(1 - \frac{N_2}{4\pi}y_2 - \frac{N_3}{4\pi}y_3\right)e^{y_2+y_3}, \tag{3.4}$$

where subscripts on f denote differentiation with respect to the appropriate y_2 or y_3 variable. This is a non-linear second order differential equation, and to our knowledge there are no general methods for obtaining a solution. However, as we explain in Appendix A, Eq. (3.4) is a Monge-Ampère equation in the two real variables y_2 and y_3. For such equations there are general existence theorems for solutions, given some fairly general boundary conditions (see Appendix A).

A crucial rôle in understanding the solutions of (3.4) is played by the choice of boundary conditions. In the following subsection we present a solution that is perturbative in the number of D7-branes. The boundary conditions that we impose imply that to lowest order the solution behave like two independent superposed stacks of D7-branes.

3.1 Perturbative solutions for two stacks of D7-branes

In order to develop a series solution to (3.4) where both N_2 and N_3 are small, we introduce an expansion parameter ϵ and define

$$n_2 = \epsilon\frac{N_2}{4\pi}, \qquad n_3 = \epsilon\frac{N_3}{4\pi}. \tag{3.5}$$

We have also absorbed the factors of 4π to simplify the subsequent expressions. Note that this expansion presupposes that both N_2 and N_3 are of the same order. In this case, (3.4) takes the form

$$f_{22}f_{33} - {f_{23}}^2 = e^{y_2+y_3}(1 - \epsilon(n_2 y_2 + n_3 y_3)). \tag{3.6}$$

We now proceed to look for a perturbative solution of the form

$$f(y_2, y_3) = f^{(0)} + \sum_{n=1}^{\infty} \epsilon^n f^{(n)}. \tag{3.7}$$

The zeroth order solution corresponds to flat space with no D7-branes, and the corresponding part of the Kähler potential is simply given by

$$f^{(0)} = e^{y_2} + e^{y_3}. \tag{3.8}$$

At the first order, we find

$$e^{-y_2} f^{(1)}_{22} + e^{-y_3} f^{(1)}_{33} = -n_2 y_2 - n_3 y_3. \tag{3.9}$$

In general, solutions to this equation are only determined up to choice of boundary conditions. However, at the linear order, this can be viewed as a sum of two equations—one for each D7 stack. Hence we choose a simple solution of the form

$$f^{(1)} = -n_2 e^{y_2}(y_2 - 2) - n_3 e^{y_3}(y_3 - 2). \tag{3.10}$$

This may be viewed as defining our boundary conditions (at first order) corresponding to the desired two stack configuration.

We now demonstrate how this series expansion may be developed to higher orders. Although we have started with a non-linear equation, the perturbative expansion yields linear equations at each order. In particular, for $n \geq 2$, $f^{(n)}$ is determined by solving

$$e^{-y_2} f^{(n)}_{22} + e^{-y_3} f^{(n)}_{33} = -e^{-y_2-y_3} \sum_{i=1}^{n-1} \left(f^{(i)}_{22} f^{(n-i)}_{33} - f^{(i)}_{23} f^{(n-i)}_{23} \right), \qquad (n \geq 2). \tag{3.11}$$

Note that the linear operator $L = e^{-y_2}\partial_2^2 + e^{-y_3}\partial_3^2$ separates, allowing a solution to be obtained through separation of variables. However we will not pursue this method here, but will instead develop the next few orders by inspection. The second order equation becomes

$$e^{-y_2} f^{(2)}_{22} + e^{-y_3} f^{(2)}_{33} = -(n_2\, n_3) y_2\, y_3, \tag{3.12}$$

demonstrating that y_2 and y_3 become mixed at this order. This equation has a solution of the form

$$f^{(2)} = ae^{y_2}\, y_3(y_2 - 2) + be^{y_3}\, y_2(y_3 - 2), \qquad a + b = -n_2\, n_3. \tag{3.13}$$

Of course, this solution is not unique. However, it maintains the structure of the lower order terms, namely $f \sim e^{y_2} f_2(y_2, y_3) + e^{y_3} f_3(y_2, y_3)$. In order to maintain symmetry between the two stacks, we take $a = b = -(n_2\, n_3)/2$. Similarly, we find a solution to third order

$$f^{(3)} = -\tfrac{1}{2} n_2^2\, n_3 e^{y_2}\, y_3(y_2^2 - 4y_2 + 6) - \tfrac{1}{2} n_2\, n_3^2 e^{y_3}\, y_2(y_3^2 - 4y_3 + 6), \tag{3.14}$$

which maintains interchange symmetry between $y_2 \leftrightarrow y_3$ (with corresponding $n_2 \leftrightarrow n_3$). Thus the above mentioned structure of f as a sum of two independent terms persists to third order. It is worth remarking, however, that the solution necessarily has a different structure at fourth order in perturbation theory. The reason is that precisely at this order the term ${f_{23}}^2$ starts contributing, and therefore breaks the symmetry of the first three orders.

Collecting the above terms in the expansion, we find that the Kähler potential takes the form

$$\begin{aligned}
\mathcal{F}_{\mathrm{D7/D7'}} \;=\;& z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 \\
& - \frac{N_2}{4\pi} z_2 \bar{z}_2 \left(\log z_2 \bar{z}_2 - 2\right) - \frac{N_3}{4\pi} z_3 \bar{z}_3 \left(\log z_3 \bar{z}_3 - 2\right) \\
& - \frac{N_2\, N_3}{32\pi^2} \Big[z_2 \bar{z}_2 (\log z_2 \bar{z}_2 - 2) \log z_3 \bar{z}_3 + z_3 \bar{z}_3 (\log z_3 \bar{z}_3 - 2) \log z_2 \bar{z}_2 \Big] \\
& - \frac{N_2^2\, N_3}{128\pi^3} z_2 \bar{z}_2 \log z_3 \bar{z}_3 \left(\log^2 z_2 \bar{z}_2 - 4 \log z_2 \bar{z}_2 + 6\right) \\
& - \frac{N_2\, N_3^2}{128\pi^3} z_3 \bar{z}_3 \log z_2 \bar{z}_2 \left(\log^2 z_3 \bar{z}_3 - 4 \log z_3 \bar{z}_3 + 6\right) + \cdots .
\end{aligned} \tag{3.15}$$

Given the Kähler potential, we can easily read off the metric. We present a few entries; the rest can be recovered using the symmetries under z_2 and z_3:

$$\begin{aligned}
g_{2\bar{2}} \;=\;& 1 - \frac{N_2}{4\pi} \log z_2 \bar{z}_2 - \frac{N_2\, N_3}{32\pi^2} \log z_2 \bar{z}_2 \log z_3 \bar{z}_3 - \frac{N_2^2\, N_3}{128\pi^3} \log^2 z_2 \bar{z}_2 \log z_3 \bar{z}_3 + \cdots , \\
g_{3\bar{2}} \;=\;& -\frac{N_2\, N_3}{32\pi^2} \left(\frac{z_2}{z_3} \left(\log z_2 \bar{z}_2 - 1\right) + \frac{\bar{z}_3}{\bar{z}_2} \left(\log z_3 \bar{z}_3 - 1\right) \right) \\
& - \frac{N_2^2\, N_3}{128\pi^3} \frac{z_2}{z_3} \left(\log^2 z_2 \bar{z}_2 - 2 \log z_2 \bar{z}_2 + 2\right) - \frac{N_2\, N_3^2}{128\pi^3} \frac{\bar{z}_3}{\bar{z}_2} \left(\log^2 z_3 \bar{z}_3 - 2 \log z_3 \bar{z}_3 + 2\right) \\
& + \cdots .
\end{aligned} \tag{3.16}$$

We see that, to first order in the number of D7-branes, the metric looks like a simple superposition of two stacks of D7. This validates our implicit choice of boundary conditions imposed above. However, already at second order we see a mixing between the z_2 and z_3 planes. Note that $g_{3\bar{2}}$ signals a highly nontrivial dependence on the coordinates z_2 and z_3 through the presence of ratios of complex coordinates.

3.2 A perturbation in the relative number of D7-branes

While in the above we have focused on N_2 and N_3 both of the same order, we may alternatively consider the case where one of the stacks of D7-branes is much smaller than the other. To be concrete, we take $N_2 \ll N_3$, and consider an expansion in terms of the small parameter N_2/N_3. In this case, we define

$$n_2 = \epsilon \frac{N_2}{4\pi}, \qquad n_3 = \frac{N_3}{4\pi}, \tag{3.17}$$

so that the determining equation, (3.4), becomes

$$f_{22} f_{33} - {f_{23}}^2 = e^{y_2+y_3}((1 - n_3 y_3) - \epsilon n_2 y_2). \tag{3.18}$$

Clearly this equation may be expanded in the same manner as (3.6). However, the source terms will be rearranged at the first two orders.

At zeroth order, we take the exact solution for a single stack of N_3 coincident D3-branes

$$f^{(0)} = e^{y_2} + e^{y_3}(1 - n_3(y_3 - 2)). \tag{3.19}$$

At first order, we find that $f^{(1)}$ must satisfy

$$e^{-y_2}(1 - n_3 y_3) f^{(1)}_{22} + e^{-y_3} f^{(1)}_{33} = -n_2 y_2. \tag{3.20}$$

This has a particularly simple solution of the form

$$f^{(1)} = -n_2 e^{y_3} y_2. \tag{3.21}$$

For higher orders, the perturbative expansion has the form

$$\left(e^{-y_2}\partial_2^2 + \frac{e^{-y_3}}{1 - n_3 y_3}\partial_3^2 \right) f^{(n)} = -\frac{e^{-y_2-y_3}}{1 - n_3 y_3} \sum_{i=1}^{n-1} \left(f^{(i)}_{22} f^{(n-i)}_{33} - f^{(i)}_{23} f^{(n-i)}_{23} \right), \qquad (n \geq 2), \tag{3.22}$$

which is again amenable to separation of variables.

Collecting the terms in the Kähler potential computed above, we obtain

$$\mathcal{F} = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 - \frac{N_3}{4\pi} z_3\bar{z}_3(\log z_3\bar{z}_3 - 2) - \frac{N_2}{4\pi} z_3\bar{z}_3 \log z_2\bar{z}_2 + \mathcal{O}(N_2^2).$$

This Kähler potential results in the following metric:

$$\begin{aligned} ds^2 &= dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + \left(1 - \frac{N_3}{4\pi}\log z_3\bar{z}_3 - \frac{N_2}{4\pi}\log z_2\bar{z}_2\right) dz_3 d\bar{z}_3 \\ & \quad -\frac{N_2}{4\pi}\left(\frac{\bar{z}_3}{\bar{z}_2} dz_3 d\bar{z}_2 + \frac{z_3}{z_2} dz_2 d\bar{z}_3\right) + \mathcal{O}(N_2^2). \end{aligned} \tag{3.23}$$

Note that the main effect of turning on a second stack of D7-branes transverse to z_2 is to change $g_{3\bar{3}}$ by an expected amount proportional to $\log z_2\bar{z}_2$. What is slightly unexpected, however, is the appearance of non-diagonal terms $g_{2\bar{3}}$ at first order in the N_2 perturbation. Nevertheless, this is consistent with the expansion of the previous subsection.

4 Probe analysis

In the previous section, we have examined the Kähler structure of the background with two perpendicular stacks of D7-branes. To proceed, we would now in principle need to solve for the D3-brane warp factor, much as we did for the single stack case in section 2.2. However, we may in fact already extract some $\mathcal{N} = 1$ flavor physics without full knowledge of the warp factor by considering a perpendicular D7$'$ probe in the backreacted D3/D7 background.

Let us first reemphasize, however, that taking the F-theory expression for the dilaton, (2.15), implies that the dilaton/axion has a very different behavior in two regions of interest: near and far from the D7-branes. Near a stack of D7-branes we have an expression of the form (2.16). However, far away from the stack (that is, for large values of z) the solution of (2.15) is roughly of the form

$$\tau = \tau_0 + \frac{c}{z^{N_f}}. \tag{4.1}$$

The above expression can be obtained from (2.15) by considering an expansion near $|z| \to \infty$. Note that the radius of convergence of the above expression excludes $z = 0$. The leading term is nothing but a constant dilaton thus allowing us to recover flat space in this limit.

The general solution to (2.11) in the case of one stack of D7-branes implies that the metric entry $g_{z\bar{z}}$ is simply proportional to τ_2. The arbitrary function Ω of (2.11) was fixed in [19] by imposing modular invariance and regularity of the metric, leading to

$$g_{z\bar{z}} = e^{\Psi} = \tau_2 |\eta(\tau)|^4 |z|^{-N_f/6}. \tag{4.2}$$

In the large $|z|$ limit that we are interested in in this section, the metric becomes that of flat space with a deficit angle. Namely, for large $|z|$ the dilaton is constant to first order and therefore we have for the relevant part of the metric (defining $z = re^{i\phi}$)

$$ds_\perp^2 = r^{-N_f/6}\left(dr^2 + r^2 d\phi^2\right) = \frac{1}{(1-\frac{N_f}{12})^2}\left[d\rho^2 + \rho^2(1-\frac{N_f}{12})^2 d\phi^2\right], \tag{4.3}$$

where $\rho = r^{1-N_f/12}$. Thus the effect of the D7 is to create a deficit angle at asymptotic infinity. Due to this deficit angle we are forced to consider $N_f \leq 12$ (where $\rho = \log r$ must be taken for $N_f = 12$)[2].

To be specific, we take the orientation of the probe D7′ brane as indicated in the table below

	0	1	2	3	4	5	6	7	8	9
D7	–	–	–	–	–	–	–	–	·	·
D7′	–	–	–	–	·	·	–	–	–	–

The D3/D7 background is

$$\begin{aligned} ds^2 &= h^{-1/2} ds_4^2 + h^{1/2}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + e^{\Psi} dz_3 d\bar{z}_3), \\ C_{0123} &= -h^{-1}, \qquad \Box_{(6)} h = -(2\pi)^4 N_c e^{-\Psi} \delta^6(z). \end{aligned} \tag{4.4}$$

In the static gauge, with an Ansatz such that angular coordinate ϕ_1 in the z_1-plane is constant and the radial coordinate has a profile $\rho_1 = \rho_1(\rho_3)$, the pull-back metric on the D7′ probe becomes

$${}^*g_{ab} d\sigma^a d\sigma^b = h^{-1/2} ds_4^2 + h^{1/2}(dz_2 d\bar{z}_2 + (e^{\Psi} + \dot{\rho_1}^2) d\rho_3^2 + e^{\Psi}\rho_3^2 d{\phi_3}^2), \tag{4.5}$$

where $\dot{\rho}_1 = \frac{d}{d\rho_3}\rho_1$. The action of the D7′ probe

$$S = -\mu_7 \int d^8\sigma e^{-\Phi}\sqrt{-\det({}^*g + 2\pi\alpha' \mathbf{F})} + \mu_7 \int {}^*C_{(8)} + \frac{\mu_7}{2(2\pi\alpha')^2}\int *C_{(4)} \wedge \mathbf{F} \wedge \mathbf{F} \tag{4.6}$$

[2] Actually, two asymptotic geometries with $N_f = 12$ may be joined to provide the F-theory fibration of K3 over a compact base. However, we do not consider this case.

becomes (up to some infinite volume factor $2\pi Vol(R^6)$)

$$S = -\mu_7 \int d\rho_3 e^{-\Phi} e^{\Psi/2} \rho_3 \sqrt{e^{\Psi} + \dot{\rho_1}^2}. \tag{4.7}$$

The equation of motion for ρ_1 determines the profile of the probe brane

$$\frac{d}{d\rho_3}\left(e^{-\Phi} e^{\Psi/2} \rho_3 \frac{\dot{\rho}_1}{\sqrt{e^{\Psi} + \dot{\rho_1}^2}} \right) = 0, \tag{4.8}$$

where in the asymptotic region $e^{\Psi} = \rho_3^{-N_f/6}$ and $e^{\Phi} = \text{const}$[3]. The most general solution in the asymptotic region is given by

$$\begin{aligned} \rho_1 &= \int d\rho_3 \sqrt{\frac{C_1}{\rho_3^2 - C_1 \rho_3^{N_f/6}}} \\ &= \frac{\sqrt{C_1}}{N-12}\left((N-12)\log\rho_3 - 12 \log\left(1 + \sqrt{1 - C_1 \rho_3^{N_f/6-2}}\right)\right) + C_2 \end{aligned} \tag{4.9}$$

where C_1, C_2 are integration constants.

Since $\rho_1 = \text{const.}$ is a solution, the probe can be placed at a fixed position in the z_1 plane and therefore the probe has a flat profile. A consequence of this fact will be the absence of a quark condensate, *i.e.* $\langle Q\bar{Q}\rangle = 0$.

4.1 The spectrum of the scalar fields

By expanding the probe action in fluctuations around the flat profile solution $\rho_1 = d$, $\phi_1 = \text{const.}$, which is a solution even in the presence of a running dilaton, and identifying the transverse fluctuations X, Y with the scalar fields of the dual gauge theory, we extract their action (to leading order) as

$$S = -\mu_7 \int \sqrt{\det {}^*g}\, e^{-\Phi}(1 + {}^*g^{ab}(\partial_a X \partial_b X + \partial_a Y \partial_b Y)), \tag{4.10}$$

where

$$\begin{aligned} \sqrt{\det {}^*g} &= \rho_2 \rho_3 e^{\Psi}, \qquad e^{\Psi} = 1 - \frac{N_f}{2\pi}\log(\rho_3), \\ e^{-\Phi} &= 1 - \frac{N_f}{2\pi}\log(\rho_3). \end{aligned} \tag{4.11}$$

[3]From $\tau = \tau_\infty + \frac{c}{(\rho_3)^{N_f}} e^{-iN_f\phi_3}$, we extract the dilaton $e^{-\Phi} = \text{Im}\tau = \text{Im}\tau_\infty - \frac{c}{\rho^{N_f}}\sin(N_f\phi_3)$. Furthermore, upon integration over ϕ_3, we remain only with the constant dilaton factor in the effective action of the D7′ probe.

The equation of motion for either of the two scalar fields (collectively denoted by X) is

$$h\eta^{\mu\nu}\partial_\mu\partial_\nu X + \left[\frac{1}{\rho_2}\partial_{\rho_2}(\rho_2\partial_{\rho_2}X) + \frac{1}{\rho_2^2}\partial^2_{\phi_2}X\right] \\ +e^{-\Psi}\left[\frac{1}{\rho_3}\partial_{\rho_3}(\rho_3\partial_{\rho_3}X) + \frac{1}{\rho_3^2}\partial^2_{\phi_3}X - \partial_{\rho_3}\Phi\partial_{\rho_3}X\right] = 0, \quad (4.12)$$

where we recall that the warp factor h obeys

$$(\partial_1\partial_{\bar{1}} + \partial_2\partial_{\bar{2}})h + e^{-\Psi}\left(\frac{1}{\rho_3}\partial_{\rho_3}\rho_3\partial_{\rho_3} + \frac{1}{\rho_3^2}\partial^2_{\phi_3}\right)h = -(2\pi)^4 N_c e^{-\Psi}\delta^6(z). \quad (4.13)$$

When substituting the warp factor into (4.12) we evaluate it at a fixed position in the z_1-plane, namely $h = h(\rho_1 = d, \phi_1 = \text{const.}, \rho_2, \rho_3)$. Even though the warp function equation cannot be solved exactly, we can attempt to solve it perturbatively in N_f, the number of D7-branes. With the change of variable $e^{\Psi/2}\rho_3 = r_3$, the metric in the z_3-plane becomes

$$\begin{aligned} \rho_3 &\approx r_3\left(1 + \frac{N_f}{4\pi}\log r_3\right), \\ ds^2 &= e^{\Psi}\left(\frac{d\,\rho_3}{d\,r_3}\right)^2 d\,r_3^2 + r_3^2 d\,\phi_3^2 \\ &\approx \left(1 + \frac{N_f}{2\pi}\right) d\,r_3^2 + r_3^2 d\,\phi_3^2, \end{aligned} \quad (4.14)$$

and the corresponding Laplacian can be approximated by

$$e^{-\Psi}\partial_3\partial_{\bar{3}} \approx \left(1 - \frac{N_f}{2\pi}\right)\frac{1}{r_3}\partial_{r_3}(r_3\partial_{r_3}) + \frac{1}{r_3^2}\partial^2_{\phi_3}. \quad (4.15)$$

The effect of the D7-branes on the background, to first order, is to generate a conical deficit in the otherwise flat z_3 directions.

One way to proceed is to take full advantage of the observation that, to first order in N_f, the geometry of the six-dimensional space transverse to the D3-branes is (conformally) flat, with a conical deficit in the z_3-plane. The warp factor is then simply

the Green's function [25] $G(\vec{z},0)$ associated with this six-dimensional geometry:

$$\begin{aligned}
ds_6^3 &\approx dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dr_3^2 + r_3^2 d\varphi^2, \\
\varphi &\sim \varphi + \left(1 - \frac{N_f}{4\pi}\right) 2n\pi \equiv \varphi + 2\nu\pi, \\
G(r_3^2, \rho_2^2+\rho_1^2, \varphi) &= \frac{1}{\rho_2^2+\rho_1^2+r_3^2}\Bigg[1 \\
&\quad -2\Bigg(\frac{\sin(\varphi-\nu\pi)}{\sqrt{2\cos(2\varphi-2\nu\pi)-2}} \mathrm{arctanh}\frac{2(1+\cos(\varphi-\nu\pi))}{\sqrt{2\cos(2\varphi-2\nu\pi)-2}} \\
&\qquad - \frac{\sin(\varphi+\nu\pi)}{\sqrt{2\cos(2\varphi+2\nu\pi)-2}} \mathrm{arctanh}\frac{2(1+\cos(\varphi+\nu\pi))}{\sqrt{2\cos(2\varphi+2\nu\pi)-2}} \Bigg)\Bigg], \\
h &\approx L^4 G(r_3^2, \rho_2^2+d^2) \approx L^4 \left(1-\frac{N_f}{2}\right)\frac{1}{r_3^2+\rho_2^2+d^2}.
\end{aligned} \tag{4.16}$$

Thus to first order in N_f the effect of the D7-branes on the warp factor is a mere rescaling with $(1 - N_f/2)$. This will carry through when investigating the meson spectrum.

As we discussed earlier, the equation of motion for the two scalar fields corresponding to fluctuations in the transverse directions z_1,

$$X(r_3, \varphi, \rho_2^2+d^2) \sim X(r_3, \varphi+2\nu\pi, \rho_2^2+d^2), \tag{4.17}$$

can be written as

$$(\Box_4 + M^2 h)X = 0. \tag{4.18}$$

where we have Fourier expanded the scalar fluctuations in the directions common with the D3-branes, $X = \int d^4k e^{ik^\mu x_\mu} X(k)$, and we have identified the Casimir invariant k^2 with the four-dimensional mass $k^2 = -M^2$. We continue to expand X into Fourier modes, according to the identifications of the wedge geometry

$$X = \sum_{m,n} X_{m,n}(\rho_2, r_3) e^{im\phi_2} e^{i\nu n\varphi}. \tag{4.19}$$

By substituting into the equation of motion, we find

$$\left(\frac{1}{\rho_2}\partial_{\rho_2}\rho_2\partial_{\rho_2} + \frac{1}{r_3}\partial_{r_3} r_3 \partial_{r_3} + \frac{N_f}{2\pi r_3}\partial_{r_3} - \frac{m^2}{\rho_2^2} - \frac{n^2\nu^2}{r_3^2} - M^2 h\right) X_{m,n} = 0. \tag{4.20}$$

The next step is to make yet another change of variables, $r_3 = R\cos\alpha$, $\rho_2 = R\sin\alpha$,

and to further expand $X_{m,n}(R,\alpha)$ into the eigenfunctions of the differential operator

$$\begin{aligned}
\mathcal{L}^2 &= \frac{1}{\sin\alpha\cos\alpha}\partial_\alpha(\sin\alpha\cos\alpha\partial_\alpha) - \frac{n^2\nu^2}{\cos^2\alpha} - \frac{m^2}{\sin^2\alpha}, \\
\mathcal{L}^2\xi_\lambda(\alpha) &= \lambda\,\xi_\lambda(\alpha), \\
X(R,\alpha) &= X(R)\xi_\lambda(\alpha),
\end{aligned} \tag{4.21}$$

where $\lambda = -l(l+2)$ and $l = |m| + |n|\nu$. We have thus reduced the equation of motion of the scalar fluctuations to an ordinary differential equation for $X(R)$:

$$\left(\frac{1}{R^3}\partial_R R^3\partial_R + \frac{1}{R^2}\lambda + \frac{N_f}{2\pi R}\partial_R + M^2 h\right)X(R) = 0. \tag{4.22}$$

The square normalizable solution is

$$\begin{aligned}
X(R) &= R^{-1-\frac{N_f}{4\pi}+b}(R^2+d^2)^{-a}{}_2F_1(-a,\frac{1}{2}+b-c;-2a;R^2+d^2), \\
a &= -\frac{1}{2} + \frac{\sqrt{1+M^2(1-N_f/2)}}{2}, \qquad b = \sqrt{(l+1)^2 - N_f/(2\pi) + N_f^2/(4\pi)^2}, \\
c &= \frac{\sqrt{4+4M^2-2\pi M^2 N_f}}{\pi},
\end{aligned} \tag{4.23}$$

where we have to impose the additional constraint that the hypergeometric series terminates:

$$\frac{1}{2} + b - c = -N, \tag{4.24}$$

with N a positive integer. To first order in the number of background D7-branes, the mass spectrum of the D7′ probe scalar fluctuations is

$$\begin{aligned}
M^2 &= 4(N+|m|+|n|+2)(N+|m|+|n|+1) \\
&\quad + N_f\Bigg(2(N+|m|+|n|+2)(N+|m|+|n|+1) \\
&\quad - \frac{(3+2(|m|+|n|)+2N)(n^2+|n|+|mn|-1)}{\pi(|m|+|n|+1)}\Bigg).
\end{aligned} \tag{4.25}$$

Notice that by setting N_f to zero we recover the spectrum of the scalar fluctuations of a D7′ probe in the $\mathrm{AdS}_5 \times S^5$ derived in [9]. Holographically, the scalar fluctuations in [9] correspond to the spinless mesons of an $\mathcal{N}=2$ SYM gauge theory, whereas in our case, the mesons belong to a theory with reduced supersymmetry ($\mathcal{N}=1$) and flavor.

5 Review of the conifold as a Kähler manifold

After exploring the basic D3/D7 and D3/D7/D7′ systems, we now turn to the conifold. Here, we review the construction of the Ricci-flat metric on the conifold. We anticipate that the level of detail included here will provide a better understanding of the choices that we make in the next section when considering the inclusion of D7-branes on the conifold. We follow the presentation of [26] but attempt to make some of the computations more explicit.

We begin by noting that the conifold is defined as the following quadric in $\mathbb{C}^4$:

$$\sum_{a=1}^{4}(w^a)^2 = 0, \tag{5.1}$$

where w^a are complex coordinates. It is convenient to define a radial coordinate as

$$\sum_{a=1}^{4}|w^a|^2 = r^2. \tag{5.2}$$

The symmetries and general form of the conifold can be made manifest by writing the defining equation as:

$$\det \mathcal{W} = 0, \qquad i.e. \qquad z_1 z_2 - z_3 z_4 = 0, \tag{5.3}$$

where

$$\mathcal{W} = \frac{1}{2}\begin{pmatrix} w_3 + iw_4 & w_1 - iw_2 \\ w_1 + iw_2 & -w_3 + iw_4 \end{pmatrix} \equiv \begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix}. \tag{5.4}$$

In this case, the radial coordinate is given by

$$r^2 = \mathrm{tr}(\mathcal{W}^\dagger \mathcal{W}). \tag{5.5}$$

We now clarify the Kähler structure, with an eye toward generalizations that include non Ricci-flat metrics. If a metric on the conifold is Kähler it can be written as

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \mathcal{F}, \tag{5.6}$$

where $\mathcal{F}$ is the Kähler potential. A Kähler potential invariant under SU(2) × SU(2) will be a function of r^2 only. For such Kähler potentials, the metric can be written as

$$g_{\mu\bar{\nu}} = \mathcal{F}'\, \partial_\mu \partial_{\bar{\nu}} r^2 + \mathcal{F}'' \partial_\mu r^2\, \partial_{\bar{\nu}} r^2, \tag{5.7}$$

where prime means differentiation with respect to r^2. In terms of $\mathcal{W}$ the metric takes the form

$$ds^2 = \mathcal{F}' \mathrm{tr}\left(d\mathcal{W}^\dagger d\mathcal{W}\right) + \mathcal{F}'' |\mathrm{tr}\mathcal{W}^\dagger d\mathcal{W}|^2. \tag{5.8}$$

The Ricci tensor for a Kähler manifold is given by

$$R_{\mu\bar{\nu}} = -\partial_\mu \partial_{\bar{\nu}} \log g, \tag{5.9}$$

where $g = \det g_{\mu\bar{\nu}}$. The Ricci flatness condition is easily obtained from computing the determinant of the metric (5.7). In particular

$$g = \det g_{\mu\bar{\nu}} = \frac{1}{|w_4|^2}\left(r^2(\mathcal{F}')^3 + r^4(\mathcal{F}')^2\mathcal{F}''\right). \tag{5.10}$$

As not all coordinates are independent, one of the coordinates, say w_4, may be eliminated in favor of the other three complex coordinates, *i.e.* $w_4^2 = -w_1^2 - w_2^2 - w_3^2$.

The question of finding the Ricci flatness condition can now be rephrased as finding the Kähler potential such that (5.10) is a product of a holomorphic and antiholomorphic function (*e.g.* $|\omega_4|^2$). The simplest way to achieve this is by introducing $\gamma = r^2\mathcal{F}'$. The determinant can now be written as

$$g = \frac{1}{2|w_4|^2 r^2}(\gamma^3)'. \tag{5.11}$$

It now follows that choosing $\gamma \propto r^{4/3}$ will remove all r dependence. This yields the Kähler potential

$$\mathcal{F} = (r^2)^{2/3} = (w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3 + w_4\bar{w}_4)^{2/3}. \tag{5.12}$$

After introducing $\rho = \sqrt{\frac{3}{2}} r^{2/3}$, we obtain the standard Ricci-flat metric on the conifold:

$$ds^2 = d\rho^2 + \rho^2\left(\frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin\theta_i^2 d\phi_i^2) + \frac{1}{9}(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i)^2\right). \tag{5.13}$$

The angular variables arise from the natural $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetries of the conifold. Furthermore, they allow us to write the general solution of the defining equation (5.3) as

$$\mathcal{W} = rL_1 Z_0 L_2^\dagger, \tag{5.14}$$

where the L_i's are $SU(2)$ matrices and Z_0 is a particular solution to the defining relation; for example

$$L_j = \begin{pmatrix} \cos\frac{1}{2}\theta_j\, e^{\frac{i}{2}(\psi_j+\phi_j)} & -\sin\frac{1}{2}\theta_j\, e^{-\frac{i}{2}(\psi_j-\phi_j)} \\ \sin\frac{1}{2}\theta_j\, e^{\frac{i}{2}(\psi_j-\phi_j)} & \cos\frac{1}{2}\theta_j\, e^{-\frac{i}{2}(\psi_j+\phi_j)} \end{pmatrix}, \qquad Z_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.15}$$

Note that, as a result of the coset structure of the conifold, only the combination $\psi = \psi_1 + \psi_2$ appears in the metric (5.13).

6 D3/D7 on the conifold

The idea of considering a stack of D3-branes at the apex of the conifold was originally considered in [27] as a way to obtain $\mathcal{N} = 1$ superconformal gauge theories. This model is very attractive from a variety of viewpoints, and has been subsequently developed into one of the trademark examples of the gauge/gravity correspondence [3].

In this section we consider adding D7-branes to the Klebanov-Witten background. We explicitly develop two ways of adding D7-branes which are characterized by the method of embedding of the D7-branes.

Our approach to including backreacted D7-branes in the geometry of the conifold is rooted in preserving the Kähler structure. The supersymmetry and integrability conditions give, as before, the same condition presented in (2.11).

6.1 The $w_4 = 0$ embedding

Given the defining equation of the conifold, (5.1), a natural embedding is to place the D7-branes at one of the w's equal to a constant. Throughout this section we consider the embedding where the D7-branes are defined by $w_4 = 0$. Note, however, that we can use the SO(4) symmetry of the conifold to substitute w_4 for any other w_i. The defining equation of the conifold and the definition of its radial coordinate preserve the SO(4) symmetry. This symmetry prevents us from finding a simple Kähler potential corresponding to the inclusion of the D7-branes. In particular, it can be shown that there are no modifications of the Kähler potential (5.12) of the form $\mathcal{F} = (r^2 f(w_4, \bar{w}_4) + h(w_4, \bar{w}_4))^{2/3}$ for arbitrary functions f and h.

Being confronted with this fact rules out the most natural modification to $\mathcal{F}$. Thus, taking into account the symmetries of the conifold, we are led to consider solutions in

two variables of the form $\mathcal{F} = \mathcal{F}(s,t)$ where

$$s = w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3, \quad t = \sqrt{w_1^2 + w_2^2 + w_3^2}\sqrt{\bar{w}_1^2 + \bar{w}_2^2 + \bar{w}_3^2}. \tag{6.1}$$

We now seek a solution to Eq. (2.11) with $\mathrm{Im}\,\tau = 1 - n_f \log t$, where $n_f = N_f/(4\pi)$:

$$\det(\partial_m \partial_{\bar{n}} \mathcal{F}) = \Omega\bar{\Omega}\,\mathrm{Im}\,\tau = \left(\frac{2}{3}\right)^4 \frac{1 - n_f \log t}{t}. \tag{6.2}$$

The denominator and constant factor on the right hand side are to ensure that the Ricci-flat solution $\mathcal{F} = (r^2)^{2/3}$ works for $n_f = 0$. With the Kähler potential taken to be a function of both s and t, this equation becomes

$$\begin{aligned} &\mathcal{F}_s^3 + \mathcal{F}_s^2(s(\mathcal{F}_{ss} + \mathcal{F}_{tt} + t^{-1}\mathcal{F}_t) + 2t\mathcal{F}_{st}) \\ &\quad +(s^2 - t^2)\mathcal{F}_s(\mathcal{F}_{ss}(\mathcal{F}_{tt} + t^{-1}\mathcal{F}_t) - \mathcal{F}_{st}^2) \;=\; \left(\frac{2}{3}\right)^4 (1 - n_f \log t). \end{aligned} \tag{6.3}$$

We look for a linearized solution of the form

$$\mathcal{F}(s,t) = (s+t)^{2/3}\left[1 + n_f\mathcal{F}^{(1)} + n_f^2\mathcal{F}^{(2)} + n_f^3\mathcal{F}^{(3)} + \cdots\right]. \tag{6.4}$$

Note that the zeroth order solution naturally corresponds to the conifold in the absence of D7-branes.

We now observe that the linearization of (6.3) splits the perturbation problem into an infinite set of equations that can be schematically written as

$$\begin{aligned} L\,\mathcal{F}^{(1)} &= \frac{4}{3}\log t, \\ L\,\mathcal{F}^{(2)} &= S_2\big[\mathcal{F}^{(1)}\big] \\ \ldots &= \ldots \\ L\,\mathcal{F}^{(n)} &= S_n\big[\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(n-1)}\big], \end{aligned} \tag{6.5}$$

where S_i are functions that can be explicitly determined. The most important property is that at any order i, the function S_i depends on already determined functions of lower order in the perturbation expansion. In addition, the same second order differential operator

$$\begin{aligned} L &= (3s^2 + 2st - 2t^2)\partial_s^2 + (2st + t^2)\partial_t^2 + (2st + 4t^2)\partial_s\partial_t \\ &\quad +(9s + 6t)\partial_s + (2s + 5t)\partial_t + 4 \\ &= (s+t)^{-2/3}\begin{pmatrix}\partial_s & \partial_t\end{pmatrix}(s+t)^{2/3}\begin{pmatrix}3s^2 + 2st - 2t^2 & st + 2t^2 \\ st + 2t^2 & 2st + t^2\end{pmatrix}\begin{pmatrix}\partial_s \\ \partial_t\end{pmatrix} + 4 \end{aligned} \tag{6.6}$$

appears at every level in the perturbation expansion. Given that the operator L is linear, and given the general form of the sources, we conclude that the above system is consistent. Moreover, one can, in principle, find a solution for any given set of boundary conditions.

In general, the functions $\mathcal{F}^{(n)}$ depend on both s and t. However, it turns out that for $\mathcal{F}^{(1)}$ we can find a solution of the form

$$\mathcal{F}^{(1)} = \frac{1}{3}(\log t - 1). \tag{6.7}$$

At second order in perturbation theory the situation is radically different, as there is no solution depending only on t. The equation at second order is

$$L\mathcal{F}^{(2)} = \left(\frac{3}{2}\right)^4 \left[\frac{1}{2}\frac{s+t}{t} - \log^2 t\right]. \tag{6.8}$$

To solve the second (and higher) order equation in general, we would like to find a Green's function for the linear operator L. To do so, it is convenient to perform the following change of variables:

$$s = \rho\cos^2\theta, \qquad t = \rho\sin^2\theta. \tag{6.9}$$

In this case the operator L in (6.6) becomes

$$L = 3\rho^2\partial_\rho^2 + 11\rho\partial_\rho + 4 + (1 - \tfrac{1}{2}\sec^2\theta)\partial_\theta^2 + \tfrac{1}{2}(\cot\theta - \tan\theta(3 + \sec^2\theta))\partial_\theta. \tag{6.10}$$

A general solution to the homogeneous equation can be constructed using separation of variables. Namely we take

$$\mathcal{F}^{(2)}(\rho,\theta) = R(\rho)\chi(\theta). \tag{6.11}$$

The θ equation, $L\chi(\theta) = \lambda\chi(\theta)$, is solved by Legendre polynomials

$$\chi = P_n(-3 + 4\cos^2\theta), \quad n = \frac{\sqrt{5-\lambda} - 1}{2}. \tag{6.12}$$

Note that, because of the way s and t are defined in (6.1) the angular variable is restricted to $\frac{1}{2} \leq \cos^2\theta \leq 1$. Thus the argument of the Legendre polynomial lies in the range $[-1, 1]$. The remaining $R(\rho)$ equation is then homogeneous in powers of ρ; hence it may be solved by taking $R \sim \rho^\lambda$. This gives the general solution to the homogeneous equation of the form:

$$\mathcal{F}^{(2)} = \sum_{n=0}^{\infty} c_n^{\pm}\rho^{-\frac{2}{3}(2\pm\sqrt{1+3n(n+1)})} P_n(-3 + 4\cos^2\theta), \tag{6.13}$$

where the $c_n^\pm$ are coefficients that must be determined by boundary conditions. Of course, to obtain the actual solution to the inhomogeneous equation (6.8), we would have to proceed with an actual construction of the Green's function. Nevertheless, even without doing so, we can discern some general features of the solution. For example, the rôle of n in (6.13) can be understood as follows. Given that $\rho = s + t$ is the distance from the D3-brane, it is natural to conclude that higher terms in n correspond to deforming the gauge theory by operators with higher conformal dimension.

From the Kähler potential we can easily derive the metric to first order in the number of D7-branes, N_f. "Turning on" the D7-branes alters the $g^0_{m\bar{n}}$ conifold geometry in two ways:

$$g_{m\bar{n}} = g^0_{m\bar{n}}\left(1 + \frac{N_f}{3}(\log t - 1)\right) - \frac{N_f}{3} r^{4/3} \partial_m w^4 \partial_{\bar{n}} \bar{w}^4. \tag{6.14}$$

First, there is the rescaling of the background metric with a factor $(1+N_f(\log t - 1)/3)$. Second, we notice the presence of some off-diagonal metric components due to the last term in (6.14). More explicitly, to first order in N_f, the correction to the six-dimensional conifold geometry reads

$$\begin{aligned}
ds^2 \;=\;& \left(1 + \frac{N_f}{3}(2\log|w_4| - 1)\right)(ds^0)^2 \\
& - \frac{N_f}{3} r^{4/3}\Bigg[\frac{1}{2}(1 - \cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2\cos(\phi_1+\phi_2))dr^2 \\
& + \frac{1}{2} r(\sin\theta_2\cos\theta_1 - \cos\theta_1\sin\theta_2\cos(\phi_1+\phi_2))dr\,d\theta_1 \\
& + \frac{1}{2} r(\sin\theta_1\cos\theta_2 - \cos\theta_2\sin\theta_1\cos(\phi_1+\phi_2))dr\,d\theta_2 \\
& + \frac{1}{2} r\sin\theta_1\sin\theta_2\sin(\phi_1+\phi_2)dr\,d(\phi_1+\phi_2) \\
& + \frac{r^2}{4}\Bigg((\cos\theta_1 - \cos\theta_2)d\psi\,d(\phi_1+\phi_2) \\
& \quad + \sin\theta_2\sin(\phi_1+\phi_2)d\psi\,d\theta_1 - \sin\theta_1\sin(\phi_1+\phi_2)d\psi\,d\theta_2 \\
& \quad + \sin\theta_1\cos\theta_2\sin(\phi_1+\phi_2)d\theta_2\,d(\phi_1+\phi_2) \\
& \quad + \sin\theta_2\cos\theta_1\sin(\phi_1+\phi_2)d\theta_1\,d(\phi_1+\phi_2) \\
& \quad + (1 - \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1+\phi_2))d(\phi_1+\phi_2)^2\Bigg)\Bigg],
\end{aligned} \tag{6.15}$$

where w_4 is given by (7.13).

6.2 A $\mathbb{P}^1$-inspired holomorphic embedding

Here we consider a different embedding which is inspired by the small resolution of the conifold [26, 28]. In this embedding the D7-branes are perpendicular to a $\mathbb{P}^1$ submanifold inside the conifold. The resolution of the conifold can naturally be described in terms of the (z_1, z_2, z_3, z_4) of (5.4). In particular, resolving the conifold means replacing the equation $z_1 z_2 - z_3 z_4 = 0$ by the pair of equations

$$\begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0, \tag{6.16}$$

where the product $\lambda_1 \lambda_2 \neq 0$. Note that $(\lambda_1, \lambda_2) \in \mathbb{CP}^1$ (any pair obtained from a given one by multiplication by a nonzero complex number is also a solution). Thus (λ_1, λ_2) is uniquely characterized by the ratio $\lambda = \lambda_2/\lambda_1$ in the region where $\lambda_1 \neq 0$. Working on this patch, a solution to (6.16) takes the form[4]

$$\mathcal{W} = \begin{pmatrix} -z_3\lambda & z_3 \\ -z_2\lambda & z_2 \end{pmatrix}. \tag{6.17}$$

Thus (z_2, z_3, λ) are the three complex coordinates characterizing the resolved conifold in the patch where $\lambda_1 \neq 0$.

As usual, the conifold metric is $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathcal{F}$, where $\mathcal{F}$ is the Kähler potential. In this case, the radial coordinate defined by (5.5) takes the form

$$r^2 = \mathrm{tr}(\mathcal{W}^\dagger \mathcal{W}) = (1 + |\lambda|^2)(|z_2|^2 + |z_3|^2). \tag{6.18}$$

The Kähler potential for the conifold is simply[5]

$$\mathcal{F} = (r^2)^{2/3} = \Big[(1 + |\lambda|^2)(|z_2|^2 + |z_3|^2)\Big]^{2/3}. \tag{6.19}$$

In this parametrization it is natural to consider a modification of the Kähler potential that depends on the two variables:

$$s = |z_2|^2 + |z_3|^2, \qquad t = |\lambda|^2. \tag{6.20}$$

[4] In the region where λ_1 is allowed to be zero we have instead $\lambda_2 \neq 0$. In this case the general solution can be written as $\mathcal{W} = \begin{pmatrix} z_1 & -z_1\mu \\ z_4 & -z_4\mu \end{pmatrix}$, where $\mu = \lambda_1/\lambda_2$.

[5] Note that in this specific patch the matrix (6.17) satisfies the defining equation for the conifold. One can think of this matrix as a "solution" with $z_1 = -\lambda z_3$ and $z_4 = -\lambda z_2$.

Doing so, we find that the determinant of the metric takes the form

$$\det g_{m\bar{n}} = \left(\mathcal{F}_t\mathcal{F}_s^2 + \mathcal{F}_t\mathcal{F}_s\mathcal{F}_{ss}s - \mathcal{F}_s\mathcal{F}_{st}^2 ts + \mathcal{F}_{tt}\mathcal{F}_s^2 t + \mathcal{F}_s\mathcal{F}_{ss}\mathcal{F}_{tt}ts\right). \tag{6.21}$$

As explained previously, our task is to find a solution $\mathcal{F}(s,t)$ of equation (2.11) with the condition that when the number of D7-branes goes to zero we recover the Kähler potential of the conifold.

We now observe that the above equation is homogeneous in powers of s (as well as in powers of t). Hence, although the equation is nonlinear, in principle all s dependence could be removed by a judicious choice of $\mathcal{F} \sim s^n$ for some power n.

This suggests a separation of variables in s and t of the form $\mathcal{F} = s^n f(t)$. Eq. (6.21) becomes:

$$\det g_{m\bar{n}} = n^3 s^{(3n-2)} f(ff''t + ff' - (f')^2 t). \tag{6.22}$$

Taking $n = 2/3$ in the above gives an s independent determinant, and leaves a non-linear ODE to solve:

$$\det g_{m\bar{n}} = \left(\frac{2}{3}\right)^3 f(ff''t + ff' - (f')^2 t) = \left(\frac{2}{3}\right)^4 (1 - n_f \log(t)). \tag{6.23}$$

We have fixed $\Omega\bar{\Omega} = (2/3)^4$ by demanding that f reduces to $f(t) = (1+t)^{2/3}$ for $n_f = 0$ (*i.e.* in the absence of D7-branes). Again, we have assumed that Ω does not change when n_f is nonzero. Although the source does not respect homogeneity in t, it is still useful to perform the change of variables

$$t = \exp(y), \quad f(t) = \exp(y/3) l(y), \tag{6.24}$$

In this case, (6.23) implies that

$$l^3 \partial_y \partial_y \log(l) = (2/3)(1 - n_f y). \tag{6.25}$$

The standard conifold case can be recovered by taking $l = (2\cosh(\frac{y}{2}))^{2/3}$. Interestingly, the combination $l = (2A\cosh(\frac{y-y_0}{2A}))^{2/3}$ also gives a Ricci flat metric where the parameters are related to those discussed at length in [29]; however we take the simplest case with $A = 1$, $y_0 = 0$.

Note that, in fact, the l^3 prefactor in (6.25) may be replaced by any non-zero power of l through the replacement $l \to l^\alpha$ where α is an arbitrary non-zero constant. Of particular interest is to take $l \to l^{2/3}$, because the conifold itself, (6.19), has this 2/3 power built in. This gives the equation

$$l^2 \partial_y \partial_y \log(l) = (1 - n_f y). \tag{6.26}$$

The Ricci-flat solutions (corresponding to $n_f = 0$) are now simply $l = 2\cosh(\frac{y}{2})$. Assuming a perturbative solution to (6.26) of the form

$$l = 2\cosh(\frac{y}{2})\exp(n_f g_1(y) + n_f^2 g_2(y) + \cdots), \tag{6.27}$$

we find that at each order, we must solve an equation of the form

$$L g_n(y) = S_n[g_1, \ldots, g_{n-1}], \qquad L = 1 + (1+\cosh y)\partial_y^2. \tag{6.28}$$

Noting that L admits the homogeneous solution

$$g(y) = c_1 \tanh(\frac{y}{2}) + c_2\left(y\tanh(\frac{y}{2}) - 2\right), \tag{6.29}$$

we may construct the Green's function

$$G(y, y_0) = g_0(y) + \frac{(2 - y_0\tanh\frac{y_0}{2})\tanh\frac{y}{2} - (2 - y\tanh\frac{y}{2})\tanh\frac{y_0}{2}}{1+\cosh y_0}\theta(y - y_0), \tag{6.30}$$

where $g_0(y)$ satisfies the homogeneous equation, and must be chosen to respect boundary conditions.

At linear order in n_f, we find $Lg_1 = -y/2$, which admits a simple solution

$$g_1 = -\frac{y}{2}. \tag{6.31}$$

Taking this form of g_1, the next order equation has source $Lg_2 = -y^2/4$, and may be solved to yield

$$g_2 = -2\log(1+e^y) + \frac{y(4-y)e^y}{2(1+e^y)} + \frac{1-e^y}{1+e^y}\mathrm{Li}_2(-e^y), \tag{6.32}$$

up to the possible addition of a solution to the homogeneous equation.

We have therefore managed, in this simple case, to find a modification to the Kähler potential of the conifold accommodating the presence of D7-branes up to second order (and in principle extendible to arbitrary order) in the number of D7-branes. Interestingly, however, as we will explain in the next section, this embedding of the D7-branes seems to be in a different class from the $w_4 = 0$ one.

7 Gauge theory

Let us start by commenting on the decoupling limit. To make connection with gauge theories we first show that the supergravity solutions presented here allow for a region

where only the low energy D-brane degrees of freedom are described. This is a necessary requirement to claim a duality between our supergravity backgrounds and the gauge theories appearing in the worldvolumes of branes.

The Yang-Mills coupling of the gauge theory living on a Dp-brane is

$$g_{Dp}^2 = g_s l_s^{p-3}. \tag{7.1}$$

We want to take the string scale to zero $l_s \to 0$ keeping the D3 SYM coupling fixed $g_{D3}^2 = g_s$. This automatically means that $g_{D7}^2 \to 0$ which implies that the D7-brane dynamics decouple, leaving us with a flavor symmetry.

There are other quantities that we will keep fixed in the limit $l_s \to 0$ such as the mass of the 7-7, 3-7 and 3-3 strings. This translates into a rescaling of the radial coordinate perpendicular to the D3-brane as $u = r/l_s^2$. This decoupling limit provides us with a consistency check on the solutions we find. In particular, we will require that when turning off the number of D7-branes the metric becomes either $AdS_5 \times S^5$ for the D7 in flat space or $AdS_5 \times T^{1,1}$ for D7's on the conifold.

The gauge theory of a stack of D7's on a D3 background has been discussed in [21]. A complete account of the degrees of freedom of this system (and in general of Dp/D(p+4)) is given in the textbook [30]. One interesting fact about the D3/D7 system is that the condition for the dilaton to be small, that is for the string loop corrections to be small requires $r \ll e^{2\pi/(g_s N_f)}$. The condition to suppress the α' corrections, on the other hand, implies in the case of D3/D7 the radial distance from the D7 must satisfy $r \gg e^{-2\pi N_c/N_f}$. These two conditions have to be taken into consideration when interpreting the range of validity of the supergravity calculation. Generically, for large $g_s N_f$ we must consider only $N_c \gg N_f$. However, for very small $g_s N_f$, there is a range in which, in principle we could relax the relation between N_c and N_f.

7.1 Matching Pathologies: Naked singularities and Landau Poles

If one restricts to the form of the dilaton given in 2.16 and rewrites the metric using $z_3 = \rho e^{i\phi}$

$$ds_6^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + \left(1 - \frac{N_f}{4\pi}\log z_3 \bar{z}_3\right) dz_3 d\bar{z}_3, \tag{7.2}$$

it becomes clear that the metric has a naked singularity when

$$e^{-\Phi} = 1 - \frac{N_f}{2\pi}\ln\left(\frac{\rho_3}{\rho_0}\right), \tag{7.3}$$

vanishes, that is, at:

$$\rho_L = \rho_0 \; e^{2\pi/g_s N_f}. \tag{7.4}$$

A simple way to realize the relation to the Landau pole is through the identification

$$\frac{1}{g_{YM}^2} = e^{-\Phi}. \tag{7.5}$$

Thus, the metric singularity coincides with the point at which $g_{YM} \to \infty$. It is, therefore, natural to consider only the region with $\rho_3 \le \rho_L$.

Gauge theory dual of D3/D7/D7′

The gauge theory living on this brane configuration has been described in the literature, for example in [24, 4]. The analysis is closely related to a configuration of intersecting M5's discussed in [31]. The theory includes the $\mathcal{N} = 4$ supermultiplet plus N_f hypermultiplets Q and N_f' hypermultiplets T transforming in the fundamental representation of SU(N). A simple way to describe this theory is via the superpotential

$$W = W_{\mathcal{N}=4} + \lambda_Q Q Z_2 \tilde{Q} + \lambda_T T Z_3 \tilde{T}, \tag{7.6}$$

where Z_2 and Z_3 are two of the three chiral superfields of the $\mathcal{N} = 4$ multiplet and we are explicitly using $\mathcal{N} = 1$ notation.

The results of our supergravity analysis point to a generic appearance of nondiagonal terms in the metric. The explicit form of the metric (3.16) shows a complicated pattern of mixing the z_2 and the z_3 coordinates. This seems to suggest that, at strong coupling, the two flavor sectors mix in a complicated way due to interactions.

Gauge theory dual of D3/D7 on the conifold

D3-branes on the conifold [27] realize an $\mathcal{N} = 1$ superconformal field theory (eight supercharges) with gauge group SU(N) × SU(N). The matter content of this theory consists of chiral superfields A_i and B_i with $i = 1, 2$ transforming as $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ respectively. The fields A and B are doublets under one of the SU(2)'s in the global symmetry SU(2) × SU(2). Under this symmetry a natural superpotential to add is [27]

$$W_{conifold} = \frac{\lambda}{2} \epsilon^{ij} \epsilon^{kl} Tr A_i B_k A_j B_l. \tag{7.7}$$

As the example of $\mathcal{N} = 4$ shows, the structure of the deformation of the superpotential in the gauge theory is dictated by the geometry in the supergravity background.

In particular, note that the superpotential for the conifold is intimately related to the defining equation for the conifold (5.3) under the identification

$$z_1 = A_1B_1, \quad z_2 = A_2B_2, \quad z_3 = A_1B_2, \quad z_4 = A_2B_1. \tag{7.8}$$

To write down the superpotential corresponding to the embedding defined by $w_4 = 0$, recall that $w_4 \sim z_1 + z_2$ and therefore the natural deformation of the above superpotential is of the form

$$W = W_{conifold} + \lambda_Q Q(A_1B_1 + A_2B_2)\tilde{Q} + \lambda_{\tilde{q}} q(B_1A_1 + B_2A_2)\tilde{q}. \tag{7.9}$$

In [7] a deformation of this type was motivated using the theory of D3-branes on $\mathbb{C}^2/\mathbb{Z}_2$.

Comments on the β-functions

The exact β function for the canonical gauge coupling of $\mathcal{N} = 1$ with gauge group $SU(N)$ and F flavors is [32]

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N - F(1-\gamma)}{1 - N\frac{g^2}{8\pi^2}}, \tag{7.10}$$

where γ is the anomalous dimension of the matter fields. One of the most attractive features of the solutions describing branes on the apex of the conifold is the precise geometrical description of the NSVZ β-function [27, 33].

One very important feature of (7.10) is that it serves as a guide in determining the boundary conditions of the supergravity solution describing flavor. Namely, as describe by equation (2.14) the dilaton can be any holomorphic function of the coordinates. In particular the sign of the coefficient of the term containing the logarithm is arbitrary in the supergravity solution. In this case equation (7.10) dictates the sign.

Another important remark about (7.10) is that it also allows to understand the region of the radial coordinates where the theory is describing the flavor sector. Note that in general the dilaton goes from a constant at infinity to a logarithmic term near the D7-branes. As we will see in what follows for the case of the conifold, only the logarithmic region (decompactification limit) contributes to the β-function in the expected way.

For the Klebanov-Witten theory the assignment of anomalous dimension of $\gamma = -1/2$ to the operators $\mathrm{Tr}A_iB_j$ guarantees that the beta function is zero, which corresponds to a superconformal fixed point [27]

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} \approx 3N - 2N(1-\gamma). \tag{7.11}$$

The two gauge couplings are related to the supergravity background as [27, 3]:

$$\begin{aligned} \frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_1^2} &= \frac{\pi}{g_s}e^{-\Phi}, \\ \frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_1^2} &= \frac{1}{2\pi\alpha'}\left(\int_{S^2} B_2\right) - \pi \mod 2\pi. \end{aligned} \tag{7.12}$$

The case with probe D7-branes was considered in [7]. In the absence of B_2 field we obtain that the two couplings are the same. To read off their value we need the explicit expression for w_4 which can be obtained from (5.3), (5.14) and (5.15)

$$w_4 = \frac{i\,r}{\sqrt{2}}e^{\frac{i}{2}(\psi+\phi_1+\phi_2)}\big[\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{-i(\phi_1+\phi_2)}\big]. \tag{7.13}$$

Assume, that the renormalization energy scale is proportional to the radius $\Lambda \sim \rho$ (the perpendicular distance from the D3-branes) we have the following expression for the running of the coupling

$$\frac{\partial}{\partial\Lambda}\frac{8\pi^2}{g_{YM}^2} = -\frac{3}{2}N_f(1-\gamma_Q), \tag{7.14}$$

The value of γ_Q can be found, similarly to a computation of [7], by power counting. Given that the dimension of the superpotential (7.9) is three, and that the dimension of the fields A's and B's is 3/4 as argued in [27], we conclude that $\gamma_Q = 1/4$.

For the case of the $\mathbb{P}^1$ parametrization of the conifold we note that $\tau \sim \log\lambda$ and

$$\lambda = \tan\frac{\theta_2}{2}e^{-i\phi_2}. \tag{7.15}$$

This combination is independent of the radial coordinate ρ, therefore, keeping the natural identification of ρ with the RG energy scale Λ we conclude that this deformation of the gauge theory does not affect the β-function and therefore does not truly amount to adding flavor to the KW gauge theory.

8 Conclusions

We conclude with a list of open problems which we believe would be interesting to address. The first, and perhaps most haunting problem in the gauge/gravity correspondence is how to achieve asymptotic freedom, or in other words how to follow the

theory from strong coupling where supergravity is a good approximation into the weak coupling regime. This problem is well beyond the scope of the present paper. But it must be kept in mind in order to understand the limitations of the present work. Some direct open problems resulting from our investigation are:

• Application of our techniques to other Kähler spaces. Most prominently, the resolved and deformed conifolds.

• The inclusion of 3-form fluxes. In the case of the conifold we refer to G_3 which corresponds to including fractional branes which change the relative rank of the two gauge groups.

• An interesting open problem would be to understand the backreaction for theories that do not necessarily admit a formulation exploiting Kähler geometry. Such seems to be the case for the Maldacena-Núñez solution and the D4/D6 bound state.

• Ideally, it would be desirable to obtain the spectrum of the different modes for the class of solutions we have presented. We should point out that some analysis of the spectrum (for example of mesons) has been presented in the probe approximation [11, 9, 10, 8]. But an analysis of the fully backreacted case will certainly yield a more robust picture. In this paper we took a step further by showing how the backreacted D3/D7 solution could be used to derive the scalar (meson) spectrum of a $\mathcal{N} = 1$ gauge theory realized on the worldvolume of a D7′ probe.

To conclude, we believe we have explicitly demonstrated a viable way to construct fully backreacted supergravity backgrounds dual to four-dimensional $\mathcal{N} = 1$ SYM with flavor in the fundamental representation. We hope to return to many of the fascinating issues in the gauge/gravity correspondence with flavor in future work.

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A Comments on the Monge-Ampère equation

In this appendix we collect some known results about the Monge-Ampère equation that are relevant for the analysis in the main text. The literature on this subject is vast [34–36] and we refer the reader to those references for further details.

We start with some motivation of why Monge-Ampère equations are relevant for our analysis. We assume (four-dimensional) Poincaré symmetry and (for simplicity) turn off 3-form flux. As in section 2, one arrives at equation (2.11) for type IIB supergravity

$$\det(\partial_\mu \partial_{\bar{\nu}} \mathcal{F}) = \Omega\bar{\Omega}\,\mathrm{Im}\,(\tau). \tag{A.1}$$

To place this in the proper context of the Monge-Ampère equation, we must change the differentiation appearing above into differentiation with respect to real variables. First, we consider the "decompactification" limit of seven branes [19], where $\tau \sim iC + ia\log(z)$. We will then place our D3's close enough to the D7's so that this approximation is valid. Thus $\mathrm{Im}\,(\tau) \sim \log(z\bar{z})$ where we have chosen our origin so that $b = 0$. We then make the Ansatz that the functional dependence of the Kähler potential is $\mathcal{F}\,(\log(z_1\bar{z_1}), \ldots, \log(z_n\bar{z_n}))$, where the log's have been inserted for later convenience. This requires that Ω has the form $\Omega = \Pi z_i^{m_i}$ for any set of m's so that both left and right hand side are functions of the new variables $y_n = \log(z_n\bar{z}_n)$. Under this change Eq. (A.1) becomes:

$$\begin{aligned} \frac{\det(\partial_i\partial_j\mathcal{F})}{\Pi\exp{(y_i)}} &= \Omega\bar{\Omega}\,\mathrm{Im}\,(\tau), \\ \det(\partial_i\partial_j\mathcal{F}) &= \exp{(\Sigma(m_i+1)y_i)}\mathrm{Im}\,(\tau). \end{aligned} \tag{A.2}$$

Note that the change of variables has introduced only a product of the z's to powers. This can be absorbed into the definition of Ω, and so has no affect on the curvature. Equation (A.2) is the standard Monge-Ampère equation in real variables. We now turn to some of the main results from the study of Monge-Ampère equations.

The Dirichlet problem for Monge-Ampère equations is formulated as follows. Let Ω be a bounded strictly convex domain in $\mathbb{R}^n$ $(n \geq 2)$ defined by a C^∞ strictly convex function h on $\bar{\Omega}$ satisfying $h|_{\partial\Omega} = 0$. Given $u(x)$, a C^∞ function on $\partial\Omega$ which is the restriction to $\partial\Omega$ of a C^∞ function γ on $\bar{\Omega}$, we consider the equation:

$$\log\det(\partial_i\partial_j\phi) = f(x,\phi), \qquad \phi|_{\partial\Omega} = u, \tag{A.3}$$

where $f(x,t) \in C^\infty(\bar{\Omega}, \mathbb{R})$. The main result that we will use is an existence theorem which can be stated as follows [34–36]. The Dirichlet problem (A.3) has a unique

solution belonging to $C^\infty(\bar{\Omega})$, when $n = 2$, if there exists a strictly convex upper solution $\gamma_0 \in C^2(\bar{\Omega})$ satisfying

$$\det(\partial^2_{ij}\gamma_0) \leq \exp f(x, \gamma_0), \qquad \gamma_0|_{\partial\Omega} = u, \tag{A.4}$$

and if $f'_t(x,t) \geq 0$ for all $x \in \Omega$ and $t \leq \sup_{\partial\Omega} u$.

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